

# A Stochastic Inventory Model with Trade Credit

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Suppliers routinely sell goods to retailers on credit. Common credit terms are tantamount to a schedule of declining discounts (escalating penalties) that depend on how long the retailer takes to pay off the supplier's loan. However, issues such as which stocking policies are optimal in the presence of supplier-provided credit have been investigated only when demand is assumed deterministic. Nearly all stochastic inventory models assume either time-invariant finance charges or charges that may vary with time but not with the age of the credit. In this article we present a discrete time model of the retailer's operations with random demand, which is used to prove that the structure of the optimal policy is not affected by credit terms, although the value of the optimal policy parameter is. This is followed by a continuous time model, which leads to an algorithm for finding the optimal stock level. We also model the supplier's problem and calculate the optimal credit parameters in numerical experiments.

*Key words:* trade credit; finance and inventory models

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## 1. Introduction

Peterson and Rajan (1997, p. 661) state, "Trade credit is the single most important source of short-term external finance for firms in the United States." Similar observations have been made about firms in Europe as well (Wilson and Summers 2002, Giannetti et al. 2006). Commonly used credit terms lower a buyer's (retailer's) inventory carrying charges for a limited period of time. If the buyer does not pay in a timely fashion, these terms imply a schedule of escalating finance charge rates (Smith 1987). Reasons that trade credit is popular include (a) the seller has better information about buyers' creditworthiness, (b) the seller can better control a buyer, e.g., by threatening to cut future supplies, (c) the seller incurs smaller transaction costs when salvaging existing assets in case of default, and (d) the seller can, in effect, discriminate on prices. Note that direct price discrimination is outlawed in the United States except in some special circumstances (see Robinson-Patman Act, 15, U.S.C.A., Section 13(a)—<http://www4.law.cornell.edu/uscode/15/.html>). From the operations and marketing perspectives, trade credit improves

suppliers' sales, which helps to grow the market in the long run (Giannetti et al. 2006).

Although there is vast literature on inventory models in deterministic and stochastic environments, nearly all articles that consider the impact of trade credit terms assume no uncertainty in demand. In particular, permissible delays in payment have been the focus of many articles within the economic order quantity (EOQ) framework. Our survey presents major themes within this line of work, as well as some papers that consider stochastic models; see Maddah et al. (2004) for an extensive review.

Beranek (1967) emphasizes the importance of paying attention to credit terms when making lot sizing decisions. He presents examples in which ignoring financial considerations can lead to an infeasible stocking policy. Haley and Higgins (1973) expand on this theme and consider the problem of jointly choosing optimal order quantity and timing of payments to the supplier when demand is constant and inventory is financed with trade credit. Goyal (1985) obtains the EOQ formula when the buyer may delay payment by a prespecified number of days, which is a common

form of trade credit. Chand and Ward (1987) interpret permissible delay in payment as a price discount and obtain a slightly different formula. Both these approaches are based on the average cost criterion. Rachamadugu (1989) presents a discounted cash flow approach for the same problem.

Aggarwal and Jaggi (1995) and Jamal et al. (1997) obtain optimal order quantities when a constant fraction of items deteriorates per unit time. Silver and Costa (1998) model outdating where items have a finite shelf life after which inventory must be salvaged. A different method of providing credit fixes a specific payment date, e.g., the 20th of the month following the date when the invoice is generated. Such models are described as having “date terms” as opposed to the “day terms” models described earlier. Examples of research with date terms credit are found in Carlson and Rousseau (1989), Kingsman (1991), and Carlson et al. (1996). Other EOQ variants are presented in Huang (2004), in which the replenishment rate is finite, Kim et al. (1995), in which the retailer also sets selling price and demand is price dependent, and Chang (2004), in which supplier credit is linked to order quantity.

The literature on stochastic inventory models with trade credit is limited. Maddah et al. (2004) present an approximate method for calculating the cost function in a periodic review setting when the  $(s, S)$  (reorder level, order-up-to level) inventory policy is used. This leads to a numerical search procedure for finding the optimal parameters  $s$  and  $S$ . Robb and Silver (2004) compare four heuristic rules against the best  $(R, S)$  policy when date terms are offered and supply and demand are random. The best review interval  $R$  is chosen from the set of integer factors and integer multiples of 30, and the best order-up-to level  $S$  is obtained from an approximate cost function. Neither article considers whether  $(s, S)$  or  $(R, S)$  policy is optimal. Moreover, the proposed procedures for finding policy parameters are approximations.

There have been several recent attempts to jointly model financial and operational decisions. Notable among these are Buzacott and Zhang (2004), who model an asset-based constraint on the amount of available capital; Babich and Sobel (2004), who consider the timing of an initial public offering (IPO) jointly with operational decisions to maximize the

present value of proceeds from an IPO; Xu and Birge (2004), who consider a firm’s capital structure in a model that characterizes production input and financial policy as endogenous decision variables; and Li et al. (2003), who present a dynamic model to maximize the expected value of dividends by simultaneously choosing operational and financial decisions. In these examples, the finance charges for borrowed capital may depend on the decision period index. However, each item in stock in a particular period incurs the same finance charge, regardless of how long ago the retailer first obtained the necessary credit. Other related works include Lederer and Singhal (1994), who show that financial considerations play a significant role in manufacturing investment choices; Birge (2000), who advocates the use of real options as a means of incorporating risk in capacity-planning models; and Gupta and Gerchak (2002), who model the impact of operational synergies in the valuation of targets in merger/acquisition cases.

In light of the lack of research on the operational impact of trade-credit terms when demand is random, we focus on the following questions. What types of stocking policies are optimal? How should the retailer determine the stocking levels and the supplier choose the credit terms?

We first present a discrete time model of the retailer’s operations. The model is used in §3 to prove that a base-stock-ordering policy is optimal under very general credit terms, according to which the retailer’s finance cost rate is a nondecreasing function of the time it takes to sell an item. Section 4 contains a continuous time model, which leads to an algorithm for choosing the optimal stock level. In §5, we explore through numerical experiments the effect on the supplier’s profit when certain credit terms are dictated by industry norms. Section 6 provides concluding remarks. All proofs are either in the appendix or in a companion online appendix.

## 2. Formulation

We model the retailer’s expected profit maximization problem as a discrete time Markov decision process (MDP). The retailer has an opportunity to order/receive shipments once in each decision epoch, which we refer to as a “day” for convenience. The retailer uses supplier-provided credit to finance

stock purchases. It will shortly become clear that when viewed from the retailer's perspective, supplier financing subsumes the option to borrow from the bank. There are no transaction costs associated with placing orders or making payments. The retailer remits the wholesale price to the supplier in each period after sales. The supplier applies these remittances to the oldest credit first. In addition, the retailer is obligated to pay finance charges on borrowed capital in each planning period. Such practices are common when retailers carry expensive spare parts or expensive products such as automobiles. We assume that the retailer chooses a stocking policy to maximize the long-run expected discounted profit, where  $\beta$  is the discount rate.

Time is indexed by  $t$ , per unit contribution margin is denoted by  $p$ , wholesale price by  $w$ , and the length of the selling season by  $n$ . That is, the retail price is  $p + w$ . Daily demand is generated by a sequence of independent, integer-valued and nonnegative random variables  $D_t$ , where  $E(D_t) < \infty$ . The holding cost of each item has two components: The physical cost of storing inventory (including warehousing and handling costs) is  $h$  per unit per period. The daily finance cost depends on the item's *shelf age*, which refers to the amount of time that an item has spent in the retailer's store. For an item whose current age is  $j$ ,  $j < \kappa$  and the finance cost rate is  $\alpha_j$ . All items with current age  $\kappa$  or more days are charged  $\alpha_\kappa$ .

From the retailer's perspective, the maximum finance charge rate cannot exceed its market (bank) borrowing rate  $\alpha_R$ . If the supplier charges more, the retailer will switch to borrowing from the bank and pay off the supplier. That is,  $\alpha_\kappa \leq \alpha_R$  must hold. Consistent with common practice, we assume that  $\alpha_j \leq \alpha_{j+1}$ , for  $j = 1, \dots, \kappa - 1$ . These arguments imply that, from the retailer's point of view, there is no difference between supplier and bank financing options once  $\alpha_j = \alpha_R$  is reached. Different financing options do affect the supplier.

Our model is motivated by the example of retailers that sell products made by a single supplier, e.g., automobile dealers that sell a single brand of cars. All goods are obtained on credit, and products are ordered at fixed intervals. At such points, sales can be observed by the manufacturer, which demands full payment for goods the retailer has already sold. The

retailer removes the cash equivalent of the profit from each cycle of inventory operations to pay dividends or as owner's profit. The supplier establishes a credit account for the retailer and keeps track of how much credit is outstanding from each of the previous periods. Interest charges are different for loans of different vintages. Specifically, interest charges for a loan are nondecreasing in the amount of time elapsed since that loan was obtained by the retailer (see Brealey et al. 2006 for additional discussion on different credit management strategies that the suppliers use).

At the beginning of each decision epoch (i.e., start of a planning period), the retailer checks on-hand inventory, backorder, and on-order inventory levels, and orders  $q_t \geq 0$  units. We assume that the supplier can deliver each order exactly  $\mu \geq 1$  planning periods later, where  $\mu = 1$  means that orders are delivered at the end of the period in which they are placed. (By design, the minimum lead time in our model is 1. When lead time is 0, the retailer carries no inventory and trade credit becomes irrelevant.) Shortages result in a penalty  $\pi$ . When  $\mu = 1$ , we obtain the optimal policy structure both when shortages are backordered and when they result in lost sales. Thus,  $\pi$  is either the unit backorder cost per period, or the unit cost of lost sales over and above the lost revenue. If  $\mu > 1$  and shortages result in lost sales the optimal policy does not have a simple structure even if the inventory finance cost is independent of shelf age (see §9.6.5 in Zipkin 2000). Therefore, we do not consider that problem instance here.

The system state at the start of period  $t$  is described by the triplet  $(\tilde{s}_t, \tilde{q}_{t-1}, b_t)$ , where “ $\sim$ ” is used to denote a vector. The  $\kappa$ -dimensional vector  $\tilde{s}_t$ , with components  $s_{i,t}$ ,  $i = 1, \dots, \kappa$ , tracks on-hand inventory in each of  $\kappa$  age categories. Each  $s_{i,t}$  is nonnegative and if  $i < \kappa$ , it denotes the number of items on hand that have been in stock precisely  $i$  days. However,  $s_{\kappa,t}$  denotes the number of items that were purchased at least  $\kappa$  days ago. The  $(\mu - 1)$ -dimensional vector  $\tilde{q}_{t-1} = (q_{t-\mu+1}, \dots, q_{t-1})$  keeps track of the number of items on order at the start of period  $t$ . These are outstanding orders that have been placed fewer than  $\mu$  periods ago. The current level of backorders is denoted by  $b_t$ . On-hand inventory and backorders cannot be simultaneously positive; i.e.,  $s_{i,t} \cdot b_t = 0$  for each  $i$  and each  $t$ . The retailer supplies demand by

first depleting  $s_{\kappa,t}$ , then  $s_{\kappa-1,t}$ , and so on. Put differently,  $s_{i,t}$ s satisfy the property that if  $s_{i-1,t} = 0$ , then  $s_{j,t} = 0$  for each  $j \geq i$ . In reality, it does not matter which item is supplied to the customer as long as an accurate account is maintained of the age of outstanding credit.

Next, we present two models starting with the lost-sales model, where  $\mu = 1$  is assumed. The second model considers complete backordering, but there  $\mu$  can be any positive integer. In both cases, material and financial flows occur at the end of each planning period. Specifically, at the end of period  $t$ ,  $t < n$ , supply  $q_{t-\mu+1}$  is received, demand  $D_t$  is realized, and supply-demand match as well as financial transactions occur. The terminal period has special material/financial flows, which will be described separately for each formulation.

### 2.1. The Lost Sales Model ( $\mu = 1$ )

Let  $r_t(\tilde{s}_t, q_t, d_t)$  denote the one-period profit function when the retailer has on-hand inventory  $\tilde{s}_t$  and orders  $q_t$  and the realized demand is  $d_t$ . Then, for  $t = 1, \dots, n-1$ , we have

$$r_t(\tilde{s}_t, q_t, d_t) = p \left[ \left( \sum_{m=1}^{\kappa} s_{m,t} + q_t \right) \wedge d_t \right] - h \left[ \sum_{m=1}^{\kappa} s_{m,t} + q_t - d_t \right]^+ - \pi \left[ d_t - \sum_{m=1}^{\kappa} s_{m,t} - q_t \right]^+ - w \sum_{m=1}^{\kappa} \alpha_m s_{m,t}. \quad (1)$$

The first term on the right-hand side is the contribution from sales in period  $t$ , where the amount sold is the smaller of on-hand inventory and demand. (Note that  $a \wedge b$  denotes  $\min\{a, b\}$  and  $(a)^+ = \max\{0, a\}$ .) The second term is the physical holding cost, and the third term is the shortage penalty. The last term represents the finance charges owed to the supplier. Note that because replenishment arrives at the end of a period, the finance charges are calculated based only on the on-hand inventory at the start of that period. In contrast, physical holding cost is calculated based on period-end inventory (which includes items from the most recent replenishment received), as is the convention in standard inventory models. Alternatively, we can make the physical holding cost dependent on starting inventory in each period. This does not affect the key results in this paper.

Knowing state  $\tilde{s}_t$  at the start of day  $t$  and action  $q_t$ , the state at the start of day  $(t+1)$  can be calculated

via the following transition equations

$$s_{\kappa,t+1} = (s_{\kappa,t} + s_{\kappa-1,t} - D_t)^+ \quad (2)$$

$$s_{i,t+1} = \left[ s_{i-1,t} - \left( D_t - \sum_{m=i}^{\kappa} s_{m,t} \right)^+ \right]^+, \quad \text{when } 1 < i < \kappa - 1 \quad (3)$$

$$s_{1,t+1} = \left[ q_t - \left( D_t - \sum_{m=1}^{\kappa} s_{m,t} \right)^+ \right]^+. \quad (4)$$

Let  $v_t(\tilde{s}_t)$  denote the retailer's optimal expected profit function from period  $t$  onward. Then, for  $t < n$ , the optimality equations are

$$v_t(\tilde{s}_t) = \max_{q_t \geq 0} E\{r_t(\tilde{s}_t, q_t, D_t) + \beta v_{t+1}(\tilde{s}_{t+1})\}. \quad (5)$$

At the end of the selling season, all transactions are financial in nature. No new orders are placed. However, consistent with our approach in periods  $t < n$ , all transactions occur at the end of period  $n$ . That is, the retailer still pays inventory investment charges in period  $n$ . Although we use this framework for consistency, the key results of this paper remain valid (after minor adjustments) even if we assume no finance charges in period  $n$ , i.e., if all loans are paid off at the start of period  $n$ . The terminal value function is

$$\begin{aligned} v_n(\tilde{s}_n) &= -w \sum_{m=1}^{\kappa} \alpha_m s_{m,n} - w \sum_{m=1}^{\kappa} s_{m,n} + u \left( \sum_{m=1}^{\kappa} s_{m,n} \right) \\ &= u \left( \sum_{m=1}^{\kappa} s_{m,n} \right) - w(1 + \hat{\alpha}_1) \sum_{m=1}^{\kappa} s_{m,n} \\ &\quad - w \sum_{j=2}^{\kappa} \hat{\alpha}_j \left( \sum_{m=j}^{\kappa} s_{m,n} \right), \end{aligned} \quad (6)$$

where  $\hat{\alpha}_j = \alpha_j - \alpha_{j-1}$  and we set  $\alpha_0 = 0$ . In Equation (6), the function  $u(\cdot)$  determines the salvage value of the leftover stock at the end of the selling season. The retailer's objective is to find a sequence of order quantities  $\{q_t^*\}$  that maximize the present value of its total expected profit  $v_0 = \beta v_1(\tilde{s}_1)$ . Note that the vector  $\tilde{s}_1$  is a problem parameter.

### 2.2. The Complete Backordering Model ( $\mu \geq 1$ )

The retailer tracks system state  $(\tilde{s}_t, \tilde{q}_{t-1}, b_t)$ . For  $t < n$ , the one-period reward function is the following analog of the function given in (1)

$$\begin{aligned} r_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t, q_t, d_t) \\ = p \left[ (b_t + d_t) \wedge \left( \sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} \right) \right] \end{aligned}$$



$$\begin{aligned}
& -h \left[ \sum_{m=1}^{\kappa} s_{m,t} + q_{t-\mu+1} - d_t - b_t \right]^+ \\
& -\pi \left[ d_t + b_t - \sum_{m=1}^{\kappa} s_{m,t} - q_{t-\mu+1} \right]^+ - w \sum_{m=1}^{\kappa} \alpha_m s_{m,t}. \quad (7)
\end{aligned}$$

In (7),  $\tilde{q}_{t-1} = \{q_{t-\mu+1}, \dots, q_{t-1}\}$  is the state of inventory on order (but not yet received) at the start of period  $t$ . Consistent with the lost sales model's one-period revenue function, the first term is period  $t$ 's contribution from sales; the second and third terms are the holding and shortage costs, and the fourth term is the finance charge.

The state transition equations are identical to (2)–(3) for  $s_{i,t+1}$  terms when  $i > 1$ . Additional system dynamics equations are as follows

$$s_{1,t+1} = \left[ q_{t-\mu+1} - \left( D_t - \sum_{m=1}^{\kappa} s_{m,t} \right)^+ - b_t \right]^+ \quad (8)$$

$$b_{t+1} = \left[ b_t + \left( D_t - \sum_{m=1}^{\kappa} s_{m,t} \right)^+ - q_{t-\mu+1} \right]^+ \quad (9)$$

$$\tilde{q}_t = (q_{t-\mu+2}, \dots, q_t). \quad (10)$$

At the start of period  $n$ , the retailer has  $\tilde{q}_{n-1}$  outstanding orders. We assume that these orders arrive at the end of period  $n$ , backlogs are cleared, and the retailer clears the supplier's loan. All transactions are assumed to occur simultaneously at the end of period  $n$ . This one-time special treatment allows us to terminate transactions at period  $n$ . Let  $v_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t)$  denote the retailer's optimal expected profit function from period  $t$  onward. Then the optimality equations are

$$\begin{aligned}
v_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t) = \max_{q_t \geq 0} E \{ & r_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t, q_t, D_t) \\
& + \beta v_{t+1}(\tilde{s}_{t+1}, \tilde{q}_t, b_{t+1}) \} \quad (11)
\end{aligned}$$

and

$$\begin{aligned}
v_n(\tilde{s}_n, \tilde{q}_{n-1}, b_n) & = u_1(\theta_n)^+ - u_2((-\theta_n)^+) \\
& - w(1 + \hat{\alpha}_1) \sum_{m=1}^{\kappa} s_{m,n} - w \sum_{j=2}^{\kappa} \hat{\alpha}_j \left( \sum_{m=j}^{\kappa} s_{m,n} \right) \\
& + (p + w) \left[ \sum_{j=1}^{\mu-1} q_{n-j} \wedge b_n \right] - w \sum_{j=1}^{\mu-1} q_{n-j}. \quad (12)
\end{aligned}$$

In (12), functions  $u_1(\cdot)$  and  $u_2(\cdot)$  determine the terminal salvage value and backorder penalty, respectively. The term  $\theta_n = \sum_{m=1}^{\kappa} s_{m,n} + \sum_{j=1}^{\mu-1} q_{n-j} - b_n$  denotes the *inventory position* at the start of period  $n$ . (Inventory position is defined in a similar fashion for any  $t$ . It equals net inventory when  $\mu = 1$  and on-hand inventory when  $\mu = 1$  and  $b_t = 0$ .) Since all outstanding orders arrive at the end of period  $n$ ,  $\theta_n$  is the end-of-horizon net stock level. That is,  $\theta_n$  is the leftover inventory, if positive, and net shortage, if negative. This explains the first two terms, which account for terminal salvage value and shortage penalty, respectively. The third and the fourth terms are standard finance charges paid on period  $n$  starting inventory. In addition, the retailer earns  $(p + w)[\sum_{j=1}^{\mu-1} q_{n-j} \wedge b_n]$  by satisfying any backorders and pays  $w \sum_{j=1}^{\mu-1} q_{n-j}$  for the arriving stock. The latter transaction does not incur any finance charges, since all payments are due at the end of period  $n$ . The last two terms of (12) reflect these cash flows.

Alternatively, we can construct a model in which material/financial transactions continue until the  $n' = (n + \mu - 1)$ th period, with  $q_n = q_{n+1} = \dots = q_{n'-1} = 0$ . That is, no new order is placed after period  $n - 1$ . It is easy to verify that the terminal value function is still given by an equation similar to (12), but with the difference that the index  $n$  is replaced by  $n'$ . Although we use the former approach in our analysis, both approaches lead to the same conclusion about the structure of the optimal ordering policy. The retailer's objective is to identify a sequence of optimal ordering decisions  $\{q_t^*\}$  that maximize  $v_0 = \beta v_1(\tilde{s}_1, \tilde{q}_0, b_1)$ .

Before closing this section, a few comments about the relationship of models in §§2.1 and 2.2 to standard inventory models are in order. Age-dependent finance charges induce a nonlinear holding cost. It has been observed in previous work (see, for example, Porteus 2002) that the optimal ordering policy is a base-stock policy when MDP value function is concave (convex in a cost-based formulation) in its state. In the setting described here, the state of the inventory system is multidimensional—consisting of the number of items on hand of each different age category, inventory on order, and backorder levels. The resulting value function is not jointly concave in state variables. However, if  $\alpha_j \equiv \alpha$ , the formulations in §§2.1 and 2.2 reduce to the analogous standard periodic review

inventory-control problems. A natural question therefore is whether the structure of the optimal ordering policy remains intact when trade-credit terms make the retailer's holding cost a function of the amount of time it takes to sell an item? We answer this question next.

### 3. The Optimal Ordering Policy

Our approach is to establish certain properties of the value function  $v_t(\cdot)$  and to then show that these properties are preserved when optimal action is taken at each stage. This leads to a characterization of the optimal value function and of the optimal policy. We carry out the ensuing analysis assuming that the item in question is measured in discrete units because potential applications of our model are to the sale of expensive discrete items such as automobiles. We use two classes of functions,  $\mathcal{F}$  and  $\mathcal{G}$ , defined below. We also use notation  $r^+(x) = r(x+1) - r(x)$  to denote the right difference of an arbitrary function  $r(\cdot)$  at  $x$  and  $\mathcal{Z}$  to denote the set of integers. Functions in  $\mathcal{F}$  and  $\mathcal{G}$  have certain properties that are useful in simplifying the optimal value function in Equations (5) and (11). These properties are presented in Lemma 1 in the appendix.

**DEFINITION 1.** The set  $\mathcal{G}$  contains all bounded real-valued functions  $g: \mathcal{Z} \rightarrow \mathfrak{R}$  that have the following properties. For any  $s \in \mathcal{Z}$

1.  $g(s) < \infty$
2.  $g^+(s) \leq g^+(s-1)$
3. There exists  $\eta < \infty$  such that  $g^+(s) < \eta$ .

Thus,  $\mathcal{G}$  is the analog of the set of bounded concave functions with finite right derivatives. The sum of functions in  $\mathcal{G}$  also belongs to  $\mathcal{G}$ ; i.e.,  $\mathcal{G}$  is closed with respect to the sum.

**DEFINITION 2.** The set  $\mathcal{F}$  contains all real-valued functions  $f: \mathcal{Z}^+ \rightarrow \mathfrak{R}$  that have the following properties

1.  $f(0) < \infty$
2.  $f^+(0) \leq 0$
3.  $f^+(s) \leq f^+(s-1)$  for any integer valued  $s \geq 1$ .

The set  $\mathcal{F}$  is the analog of the set of bounded and decreasing concave functions. It is straightforward to verify that  $\mathcal{F}$  is a subset of  $\mathcal{G}$  and closed with respect to the sum.

#### 3.1. The Lost Sales Model

We will show that the optimal order quantity depends only on the total on-hand inventory (regardless of

shelf age) and that a base-stock policy is optimal. This result depends on the fact that the optimal value function is a sum of functions in the sets  $\mathcal{F}$  and  $\mathcal{G}$ . The arguments of these functions are partial sums of  $s_{i,t}$ s.

**PROPOSITION 1.** If  $u(\cdot) \in \mathcal{G}$  and  $u^+(0) < w$ , then the optimal value function  $v_t(\tilde{s}_t)$  can be written as a sum of functions in the sets  $\mathcal{F}$  and  $\mathcal{G}$ . Specifically, for each  $t = n, n-1, \dots, 1$ , we have

$$v_t(\tilde{s}_t) = f_1^{(t)}(s_{\kappa,t}) + f_2^{(t)}(s_{\kappa,t} + s_{\kappa-1,t}) + \dots + f_{\kappa}^{(t)}\left(\sum_{m=1}^{\kappa} s_{m,t}\right) + g^{(t)}\left(\sum_{m=1}^{\kappa} s_{m,t}\right), \quad (13)$$

where  $f_i^{(t)} \in \mathcal{F}$  for each  $i$ , and  $g^{(t)} \in \mathcal{G}$ .

The conditions under which Proposition 1 holds have a straightforward intuitive meaning. The first condition,  $u(\cdot) \in \mathcal{G}$ , ensures that the salvage value would not increase at an increasing rate in the amount of leftover stock. The second condition,  $u^+(0) < w$ , guarantees that it is never optimal to purchase stock in period  $n-1$  only to salvage it in period  $n$ . This comes from the fact that the left-hand side of this inequality is an upper bound on the incremental benefit from having one more unit to salvage in period  $n$ ; the right-hand side is the cost of purchasing an item in period  $(n-1)$  and paying for it without incurring a finance charge. The first condition is easily satisfied when  $u(x) = u \cdot x$  is a linear function. The second condition is met when the unit salvage value is smaller than the wholesale price ( $u < w$ ). Note that these are both common assumptions in inventory models with item age-independent finance charges (see, for example, Porteus 2002) and that  $v_t(\tilde{s}_t)$  is not jointly concave in  $s_{i,t}$ s.

The first consequence of Proposition 1 is that the value function  $v_t(\tilde{s}_t)$  is bounded, being the sum of bounded functions. The second consequence, also our main result, is that the optimal policy is base-stock policy.

**THEOREM 1.** If  $u(\cdot) \in \mathcal{G}$  and  $u^+(0) < w$ , then the optimal ordering policy in period  $t$  is a base-stock policy. Specifically, if  $s_t = \sum_{m=1}^{\kappa} s_{m,t}$  is the total on-hand inventory at the start of period  $t$ , then  $q_t^* = (y_t^* - s_t)^+$ , where  $y_t^* < \infty$  is the optimal base-stock level in period  $t$ .

### 3.2. The Complete Backordering Model

The structure of the optimal value function is given in Proposition 2. Its proof can be found in the appendix, available online.

**PROPOSITION 2.** *If  $u_1(\cdot) \in \mathcal{G}$ ,  $u_1^+(0) < w$ , and  $-u_2(\cdot) \in \mathcal{F}$ , then the optimal value function  $v_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t)$  can be written as a sum of functions in the sets  $\mathcal{F}$  and  $\mathcal{G}$ . Specifically, for each  $t = n, n-1, \dots, 1$ , we have*

$$v_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t) = f_1^{(t)}(s_{\kappa,t}) + f_2^{(t)}(s_{\kappa,t} + s_{\kappa-1,t}) \\ + \dots + f_{\kappa}^{(t)}\left(\sum_{m=1}^{\kappa} s_{m,t}\right) + g^{(t)}(\theta_t), \quad (14)$$

where  $f_i^{(t)} \in \mathcal{F}$  for each  $i$ , and  $g^{(t)} \in \mathcal{G}$ .

The salvage value function  $u_1(\cdot)$  must have properties similar to the properties of  $u(\cdot)$  in Proposition 1. A linear salvage value function, i.e.,  $u_1(x) = u_1 \cdot x$ , with the unit salvage value  $u_1$  smaller than the wholesale price  $w$  satisfies these requirements. The additional requirement  $-u_2(\cdot) \in \mathcal{F}$  ensures that the cost of shortage is increasing at an increasing rate in the number of units short. This requirement helps to eliminate cases in which having a large shortage at the end of the planning horizon can be beneficial. It is also easily satisfied by a linear function  $u_2(x) = \pi_n x$ , with a per unit terminal shortage penalty  $\pi_n > 0$ .

As in §3.1, a base-stock policy is also optimal when shortages are backordered. We present this result in Theorem 2 without proof, because its proof is similar to the proof of Theorem 1.

**THEOREM 2.** *If  $u_1(\cdot) \in \mathcal{G}$ ,  $u_1^+(0) < w$ , and  $-u_2(\cdot) \in \mathcal{F}$ , then the base-stock ordering policy is optimal. Specifically, if  $\theta_t = \sum_{m=1}^{\kappa} s_{m,t} + \sum_{j=1}^{\mu-1} q_{t-j} - b_t$  is the inventory position at the start of period  $t$ , then  $q_t^* = (y_t^* - \theta_t)^+$ , where  $y_t^* < \infty$  is the optimal base-stock level in period  $t$ .*

In each model, the one-period expected reward is bounded and the demand is finite with probability one. Therefore, when parameters are stationary and demand in each period is independent and identically distributed (i.i.d.), the value function iterations in (5) and (11) converge in the limit as  $n \rightarrow \infty$ . (Formal arguments can be constructed using an approach outlined in Chapter 6 of Puterman 1994.) For example, if  $v(\tilde{s}, \tilde{q}, b) = \lim_{t \rightarrow \infty} v_t(\tilde{s}_t, \tilde{q}_{t-1}, b_t)$ , then Proposition 2 and Theorem 2 also apply to the value function

$v(\tilde{s}, \tilde{q}, b)$ . The base-stock policy now has a fixed order-up-to level  $y^*$ .

Why in each case does the optimal ordering decision depend only on  $\theta_t$  (which in turn is a function of  $\sum_{m=1}^{\kappa} s_{m,t}$ ) and not on the distribution of inventory by shelf age? A clue to understanding this result on an intuitive level is provided by the one-period reward functions in Equations (1) and (7).

Note that the first three terms of these functions constitute the standard tradeoff between sales revenue on the one hand and the cost of either overage or underage, on the other hand. Such tradeoffs are common in inventory models. Demand, physical holding cost, and shortage cost are shelf age independent, so these terms depend on  $\tilde{s}_t$  only through  $\sum_{m=1}^{\kappa} s_{m,t}$ , as they do in standard inventory models. The last term captures the finance charges, which do depend on the age of each item in stock. However, given total stock level at the start of a period, the amount of time that a newly arrived item spends on the shelf is independent of the age distribution of on-hand stock. Therefore, credit terms do not affect the type of policy that is optimal. They do affect the optimal choice of  $\{q_t\}$  through the target total stock levels. The key assumption that leads to this behavior is that customers perceive no difference among items based on their shelf ages.

In the broader context of finance and operational decisions, Theorems 1 and 2 suggest that although finance and operations managers may continue to use base-stock-ordering policies, the calculation of base-stock levels should be adjusted to match credit terms. Discussion of the interactions between production and financial decisions in other contexts can be found in Ravid (1988).

Suppliers may place a limit on the total amount of stock a retailer may order/obtain on credit, where both the current level of credit and the maximum credit are defined with respect to the retailer's inventory position. Let  $\hat{y}_t$  be the maximum period  $t$  inventory position permitted by the financing constraint. Then the retailer should order the smaller of  $q_t^*$  or  $\hat{q}_t = (\hat{y}_t - \theta_t)^+$ . This simple adjustment works because for each  $t$ ,  $g^{(t)}(\cdot)$  is the discrete analog of a bounded concave function (see Equation (14)).

## 4. Policy Parameter Optimization

Although the discrete time framework described in §3 is appropriate for identifying the structure of

the optimal policy, the multidimensional MDP is computationally intractable for finding the optimal base-stock level. For this reason, we develop a continuous time model for policy parameter evaluation. Our approach parallels the approach used in previous studies. For example, Zipkin (2000) in Chapter 9 uses a discrete time model for proving the structure of optimal policy, whereas Chapter 6 of Zipkin (2000) contains a continuous time model for policy parameter optimization.

There are two key differences between a continuous time model and our approach in §3. First, in this situation, the retailer observes every demand and practices item-for-item replenishment, if she controls inventory by keeping a fixed base-stock level  $y^*$ . Second, the continuous time model maximizes average expected profit (per unit time), whereas the discrete time model is based on the discounted total expected profit criterion. See Porteus (1985, 2002) for discussions of the care necessary when approximating discounted reward models with policies computed for average reward models. Note that similar differences also exist among inventory models that have been proposed when the finance rate is shelf-age independent.

We assume that all parameters are stationary and focus only on the analog of the model in §3.2 because in a continuous time framework, the retailer may choose  $y^* > 0$  to protect against stockouts whenever the replenishment lead time is strictly positive. In §§2 and 3, we had assumed that  $\{D_t\}$  is a sequence of independent random variables. Now the retailer's demand is assumed to be Poisson distributed, with mean  $\lambda$ . The need for this assumption is explained in the ensuing analysis.

The trade-credit terms are specified by the process  $\alpha = \{\alpha(t) : t \geq 0\}$ . Knowing  $\alpha$ , the retailer chooses an appropriate base-stock level  $y^*$ . Our goal in this section is to determine a procedure for computing  $y^*$ . As in the standard stochastic inventory model (i.e., when the inventory finance rate is constant),  $y^* = \arg \max_{y \geq 0} \{\Pi_R(y | \alpha) = \lambda p - \Pi_C(y | \alpha)\}$ , where  $\Pi_R(\cdot)$  and  $\Pi_C(\cdot)$  denote the retailer's long-run average profit and cost functions, respectively. Clearly, the main task is to compute  $\Pi_C(y | \alpha)$ , the cost of demand uncertainty. This cost has two components: the cost of holding inventory and the cost of shortages. Calculation of the latter is identical to the standard inventory model.

However, the cost of carrying inventory depends on the amount of time it takes to sell each item from the moment that item arrives on the retailer's shelf. To calculate  $\Pi_C(\cdot)$ , we need to determine the limiting distributions of the following system performance measures:  $I_y(t)$ , the on-hand inventory level;  $B_y(t)$ , the backorder level; and  $A_y(t)$ , the item shelf age. As in the standard inventory model, it is possible to show that for each fixed base-stock level  $y$ , the limiting distributions of the  $I_y(t)$ ,  $B_y(t)$ , and  $A_y(t)$  exist as  $t \rightarrow \infty$  (see Zipkin 2000). We use notations  $I(y)$ ,  $B(y)$ , and  $A(y)$  to denote random variables whose distributions are identical to the limiting distributions of  $I_y(t)$ ,  $B_y(t)$ , and  $A_y(t)$ , respectively. Similarly, random variables  $IN(y)$  and  $IO$  have the limiting distributions of net inventory and inventory on order, respectively. Note that the distribution of inventory on order is independent of  $y$ , as explained below.

Consider first the cost of shortages. This can be calculated using the following fundamental relationship among the base-stock level, net inventory, and inventory on order

$$IN(y) = y - IO = y - D(\mu), \quad (15)$$

where  $D(\mu)$ , the number of units demanded in an interval of length  $\mu$ , is Poisson distributed with parameter  $\lambda\mu$ . The above relationship can be explained by observing that the net inventory deviates from  $y$  only because of outstanding orders, and that, at any given time, the number of outstanding orders are all those placed within the previous  $\mu$  time units. A backorder occurs only when the number of outstanding orders exceeds  $y$ . Therefore,

$$\begin{aligned} E[B(y)] &= \sum_{m=0}^{\infty} mP(IN(y) = -m) = \sum_{m=0}^{\infty} mP(D(\mu) = y + m) \\ &= E[(D(\mu) - y)^+], \end{aligned} \quad (16)$$

and the average shortage cost is  $\pi E[B(y)]$ .

To compute the inventory holding cost, we first work out the details for a tagged item whose shelf age is known to be  $\tau$ . Define  $a(\tau) \doteq \int_{t=0}^{\tau} \alpha(t) dt$  as the finance charge incurred on each dollar borrowed from the supplier to place the tagged item on the shelf. Then this item incurs a total finance charge of  $wa(\tau)$ . The physical holding cost incurred during  $\tau$  is simply  $h\tau$ . The expected holding cost per item is therefore



$E[h\tau + wa(\tau) | A(y) = \tau] = hE[A(y)] + wE[a(A(y))]$ . Since there are  $\lambda$  arrivals per unit time, the average holding cost per unit time is obtained by multiplying the per item holding cost with  $\lambda$ . Putting it together, we have

$$\Pi_C(y | \alpha) = \lambda \{ hE[A(y)] + wE[a(A(y))] \} + \pi E[B(y)], \quad (17)$$

where, from Little's law,  $E[A(y)] = \lambda^{-1}E[I(y)]$ . The expected on-hand inventory can be calculated by using arguments similar to (16). In particular,  $E[I(y)] = \sum_{m=0}^y mP(IN(y) = m) = E[(y - D(\mu))^+]$ . Finally,

$$\begin{aligned} y^* &= \arg \max_{y \geq 0} \{ \lambda p - \Pi_C(y | \alpha) \} \\ &= \arg \min_{y \geq 0} \{ hE[A(y)] + \pi E[B(y)] + \lambda wE[a(A(y))] \}. \end{aligned} \quad (18)$$

Equation (18) reveals two additional challenges before we can develop an efficient technique for finding  $y^*$ . First, we need a method for computing  $E[a(A(y))]$ , which requires characterization of the distribution of  $A(y)$ . Second, we need to show that  $E[a(A(y))]$  has monotone increasing differences in  $y$ . Propositions 3 and 4 present the required results.

**PROPOSITION 3.** For a fixed base-stock level  $y$ , let  $F_{A(y)}$  denote the cumulative distribution function of  $A$ . Then, for any  $t \geq 0$ ,

$$F_{A(y)}(t) = 1 - e^{-\lambda(\mu+t)} \sum_{m=1}^y \frac{(\lambda\mu)^{y-m}}{(y-m)!} \left( \sum_{i=0}^{m-1} \frac{(\lambda t)^i}{i!} \right). \quad (19)$$

**PROPOSITION 4.** For a fixed base-stock level  $y$ , let  $\phi(y) = E[a(A(y))]$ . If  $\alpha(t)$  is increasing in  $t \geq 0$ , then  $\phi^+(y) \geq 0$  and  $\phi(y+1) - \phi(y) \geq \phi(y) - \phi(y-1)$ , for any  $y \geq 1$ .

Proposition 3 follows from the fact that if an arriving replenishment observes  $m \geq 0$  items in store, then its shelf age distribution is the sum of  $m+1$  exponential demand inter arrival times. Unconditioning on  $m$  gives rise to the distribution in (19). The proof of Proposition 3 also underscores the need for Poisson-distributed demand arrivals. If demand distribution is arbitrary, a replenishment arrival epoch is not necessarily an arbitrary observation epoch, and therefore the above argument will not hold. Proposition 4 comes from the fact that  $A(y)$  is stochastically increasing in  $y$  and  $a(t)$  is increasingly convex in each  $t$ .

**THEOREM 3.** If  $\alpha(t)$  is increasing in  $t \geq 0$ , the optimal base-stock level can be computed as follows

$$y_R(\alpha) = \begin{cases} 0 & \text{if } \Pi_C(1 | \alpha) > \Pi_C(0 | \alpha), \\ \max\{y | \Pi_C(y | \alpha) - \Pi_C(y-1 | \alpha) \leq 0\} & \text{otherwise.} \end{cases} \quad (20)$$

We do not provide a proof of Theorem 3; it follows easily from the first-order optimality condition (Luenberger 1984).

Equation (20) can be further simplified, but unfortunately, it does not yield a closed-form expression for  $y_R$ . However, we show that favorable credit terms lead to the retailer stocking more; see Proposition 5. An immediate corollary of Proposition 5 is that both the retailer and the supplier benefit from trade credit. For this purpose, suppose processes  $\alpha_1$  and  $\alpha_2$  are two credit schemes such that  $\alpha_1(t) \leq \alpha_2(t)$  for all  $t \geq 0$ , and let  $y_R(\alpha_i)$  denote the corresponding order quantities that maximize the retailer's expected profit.

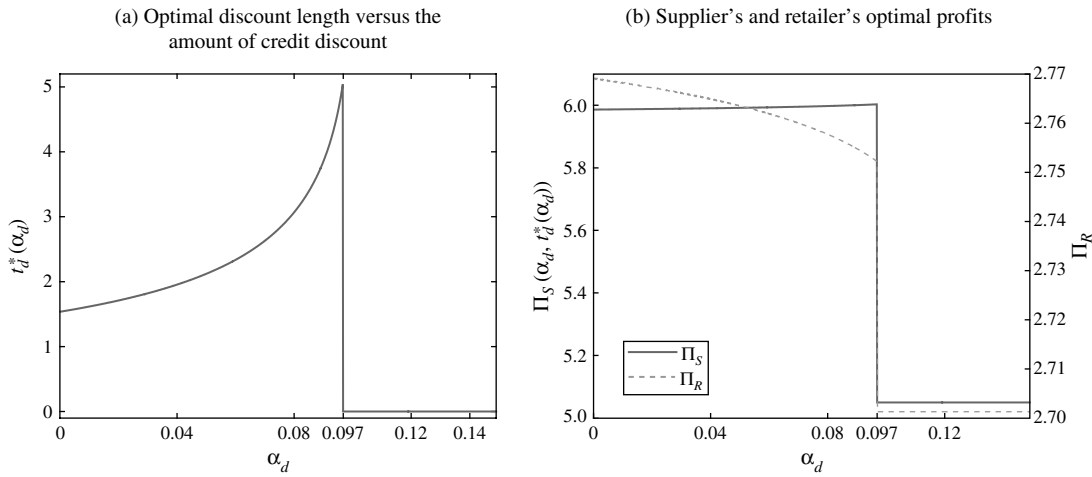
**PROPOSITION 5.** Given  $\alpha_i$  such that  $\alpha_1(t) \leq \alpha_2(t)$  for all  $t \geq 0$ ,  $y_R(\alpha_1) \geq y_R(\alpha_2)$ .

From (17), we observe that the retailer's cost decreases with favorable credit terms and when all other parameters remain fixed. Therefore, the retailer also benefits from choosing the optimal stocking level under trade credit. The supplier sets credit terms and cannot be worse off as a result of offering credit.

## 5. Examples

We report results from two examples in which the supplier offers a discounted finance rate  $\alpha_d$  if the retailer pays off its loan by  $t_d$ , and the retailer's market rate is  $\alpha_R$  thereafter. This structure of credit terms is common practice. Given the structure, our examples demonstrate how the supplier's choice of discount length and expected profit vary when the discount rate  $\alpha_d$  is exogenously determined. The discount rate could be dictated, for example, by relevant industry norms. See Kirkman (1979) and Wilson and Summers (2002) for further discussion on suppliers' ability to set arbitrary credit terms. In both examples, the supplier is assumed to have make-to-order manufacturing operations, no finished goods inventory, and sufficient production capacity to produce and deliver each order

Figure 1 The Effect of Credit Discount



after a lead time of  $\mu$ . We also assume that every exogenous demand triggers a production order.

The supplier borrows at rate  $\alpha_s$  to finance its operations. Its per unit production cost is  $c$ , and its share of the backorder cost is  $\pi_s$ . A supplier that offers terms  $(\alpha, t)$  and whose action in turn generates a response  $y$  from the retailer has the following expected profit per unit time

$$\begin{aligned} \Pi_S(\alpha, t, y) &= (w - c)\lambda - \pi_s E[B(y)] - c\alpha_s \lambda \mu \\ &\quad + \lambda \{ wE[a(t \wedge A(y))] - c\alpha_s E[t \wedge A(y)] \}. \end{aligned} \quad (21)$$

In this equation, the first term is the rate of contribution from sales, the second term is the expected shortage penalty per unit of time, the third term accounts for cost of average work-in-process inventory during the lead time  $\mu$ , and the last term accounts for the difference between the amount that the supplier collects in finance charges from the retailer (per unit of time) and the charges that it incurs to finance inventory. We assume that the supplier, like the retailer, removes the cash equivalent of the profit from each cycle of inventory operations to pay dividends or as owner's profit. For each  $\alpha_d$ , we use  $t_d^*(\alpha_d)$  to denote the corresponding optimal length of the discount period.

Although an alternative sales contract in which the supplier demands full payment at  $t_d$  is also common, as long as  $c\alpha_s < w\alpha_R$  (which we assume), the supplier earns a greater profit by extending the loan beyond  $t_d$ .

To the retailer, both options lead to the same finance rate of  $\alpha_R$  beyond  $t_d$ . Therefore, we do not consider the case where the retailer borrows from the supplier first and then from the bank.

Examples reported in Figures 1(a) and 1(b) are based on the following data:  $\lambda = 1$ ,  $h = 2$ ,  $w = 20$ ,  $p = 5$ ,  $c = 10$ , and  $\pi_R = \pi_S = 1$ ,  $\mu = 3$ ,  $\alpha_R = 0.15$ , and  $\alpha_S = 0.1$ . We vary  $\alpha_d$  from 0 to  $\alpha_R$  and study its effect on the choice of  $t_d^*$  and on the expected profits of the supplier and the retailer. In Figure 1(a), a smaller discount (greater  $\alpha_d$ ) at first implies a longer discount period. This is satisfying on an intuitive level. Up to a point, the retailer can make up for a lower discount by having more time to pay the supplier's loan. Eventually, however, the discount period covers the entire shelf age of most items, and an even longer  $t_d$  does not adequately compensate for a higher  $\alpha_d$ . At this point, faced with higher holding costs, the retailer lowers its stocking level and the supplier minimizes its finance cost by reducing the discount period to zero.

Figure 1(b) plots the supplier's and the retailer's expected profit for the same range of  $\alpha_d$  values. As explained, the supplier's profit drops once  $\alpha_d$  reaches a level at which the retailer's optimal base-stock level drops. Compared to its best profit, which is realized at  $\alpha_d = 0.097$ , the supplier's expected profit drops by 18.9% when  $\alpha_d = \alpha_R$ . The corresponding drop in the retailer's expected profit is 2.5%. Thus, the supplier benefits much more than the retailer by offer-

ing a credit discount. This is expected because the supplier sets the credit parameters to maximize its individual expected profit.

## 6. Conclusion

Trade credit is by far the most common method suppliers use to subsidize retailer holding costs. This article examined the impact of common credit terms on inventory decisions when demand is random. We showed that the base-stock inventory control policy continues to be optimal under an increasing schedule of finance charges related to payment date. Finance and accounting literature points out that in many industry sectors, credit terms are set in a certain manner for historical reasons. More importantly, small firms may not have the market power to change these terms (Kirkman 1979, Wilson and Summers 2002). Our analysis shows that to a large extent, the suppliers can correct the inefficiency introduced by the use of either the industry-standard discount period length or the industry-standard discount rate by adjusting the other parameter appropriately. These parameters serve as substitutes in terms of their effect on the retailer's stocking levels.

There are many opportunities for future work. For example, if the retailer has alternate uses of capital that earn more than the cost of borrowing from the supplier, it may strategically choose when to pay back the supplier's loan. This is particularly relevant when the retailer is credit constrained and purchases items from many suppliers with different credit terms. Similarly, issues such as the structure of the optimal ordering policy when the retailer incurs a fixed ordering cost, or when the supplier offers common terms to multiple retailers that differ in their size/credit worthiness, remain largely unexplored. Other avenues concern the nature of supplier-retailer interactions when the former sells multiple products to the same retailer and the effect of retailer (and/or supplier) competition. The authors plan to pursue these avenues in the future.

### Electronic Companion

An electronic companion to this paper is available on the *Manufacturing & Service Operations Management* website (<http://msom.pubs.informs.org/ecompanion.html>).

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## Appendix

PROPERTIES OF  $\mathcal{F}$  AND  $\mathcal{G}$ . Let the notation “ $\doteq$ ” denote a defining equality. Then each statement in Lemma 1 can be verified by checking that the resulting functions continue to satisfy the appropriate set inclusion properties.

LEMMA 1. Suppose  $f$  and  $g$  are arbitrary functions in sets  $\mathcal{F}$  and  $\mathcal{G}$ , respectively.

- Let  $g_1(s) \doteq \max_{a \geq 0} \{g(s+a)\}$ . Then  $g_1 \in \mathcal{G}$ .
- For any nonnegative and integer-valued random variable  $X$ , which is independent of  $s$  and has a finite mean, let  $f_1(s) \doteq E[f((s-X)^+)]$ . Then  $f_1 \in \mathcal{F}$ .
- Let  $g_2(s) \doteq E[(p+\pi)(s \wedge X) + g(s-X)]$ , where  $X$  is as defined above. Then  $g_2 \in \mathcal{G}$ .
- Let  $g_3(s) \doteq g(s) + f(s)$ . Then  $g_3 \in \mathcal{G}$ .
- $\beta g \in \mathcal{G}$  and  $\beta f \in \mathcal{F}$ , where  $0 < \beta < 1$ .

A proof of Lemma 1 can be found in the appendix (online).

PROOF OF PROPOSITION 1. We prove Proposition 1 by induction, starting with  $t = n$ . Let  $f_\kappa^{(n)}(s) \doteq u(s) - w(1 + \hat{\alpha}_1)s$  and  $f_{\kappa-j}^{(n)}(s) \doteq -w\hat{\alpha}_{j+1}s$ , where  $1 \leq j < \kappa$ . Since  $u(s) \in \mathcal{G}$  and  $-w(1 + \hat{\alpha}_1)s \in \mathcal{F}$ , it is clear from part (d) of Lemma 1 that  $f_\kappa^{(n)}(s) \in \mathcal{G}$ . Furthermore, since  $f_\kappa^{(n)+}(0) = u^+(0) - w(1 + \hat{\alpha}_1) < 0$ , it follows that  $f_\kappa^{(n)}(s) \in \mathcal{F}$ . Similarly,  $f_j^{(n)}$ , for  $j < \kappa$ , are decreasing linear functions of their arguments. That is, each function  $f_j^{(n)}$  belongs to the set  $\mathcal{F}$ . Next, we define  $g^{(n)}(s) \doteq 0$  for all  $s$ , and on writing terms on the right-hand side of (6) in terms of functions  $f_j^{(n)}$  and  $g^{(n)}$ , we note that

$$v_n(\bar{s}_n) = f_1^{(n)}(s_{\kappa,n}) + f_2^{(n)}(s_{\kappa,n} + s_{\kappa-1,n}) \\ + \dots + f_\kappa^{(n)}\left(\sum_{m=1}^{\kappa} s_{m,n}\right) + g^{(n)}\left(\sum_{m=1}^{\kappa} s_{m,n}\right).$$

That is, the terminal value function has the desired structure of Equation (13).

Assume that the value function has the desired structure for time indices  $t+1, t+2, \dots, n$ . Using this induction hypothesis, we will prove that the structure is preserved in  $v_t$ . After straightforward algebraic manipulations, Equation (5) can be rewritten as

$$v_t(\bar{s}_t) = -w \sum_{i=1}^{\kappa} \hat{\alpha}_i \left(\sum_{m=i}^{\kappa} s_{m,t}\right) - \pi E(D_t) \\ + \max_{q_t \geq 0} E \left\{ (p + \pi) \left[ \left(\sum_{m=1}^{\kappa} s_{m,t} + q_t\right) \wedge D_t \right] \right. \\ \left. - h \left(\sum_{m=1}^{\kappa} s_{m,t} + q_t - D_t\right)^+ + \beta v_{t+1}(\bar{s}_{t+1}) \right\}. \quad (22)$$

From the induction hypothesis,  $v_{t+1}$  can be written as the sum of functions of partial sums of  $\{s_{m,t+1}\}$ . Also, from Equations (2)–(4), we have

$$\sum_{m=i}^{\kappa} s_{m,t+1} = \left( \sum_{m=i-1}^{\kappa} s_{m,t} - D_t \right)^+ \quad \text{for } i=2, \dots, \kappa \quad \text{and} \quad (23)$$

$$\sum_{m=1}^{\kappa} s_{m,t+1} = \left( q_t + \sum_{m=1}^{\kappa} s_{m,t} - D_t \right)^+. \quad (24)$$

Using the induction hypothesis and Equations (22)–(24), we get:

$$\begin{aligned} v_t(\tilde{s}_t) = & -w \sum_{i=1}^{\kappa} \hat{\alpha}_i \left( \sum_{m=i}^{\kappa} s_{m,t} \right) - \pi E(D_t) \\ & + \beta E \left\{ f_1^{(t+1)} \left( \left( \sum_{m=\kappa-1}^{\kappa} s_{m,t} - D_t \right)^+ \right) \right. \\ & \quad \left. + \dots + f_{\kappa-1}^{(t+1)} \left( \left( \sum_{m=1}^{\kappa} s_{m,t} - D_t \right)^+ \right) \right\} \\ & + \max_{q_t \geq 0} E \left\{ (p + \pi) \left[ \left( \sum_{m=1}^{\kappa} s_{m,t} + q_t \right) \wedge D_t \right] \right. \\ & \quad \left. - h \left( \sum_{m=1}^{\kappa} s_{m,t} + q_t - D_t \right)^+ \right. \\ & \quad \left. + \beta f_{\kappa}^{(t+1)} \left( \left( \sum_{m=1}^{\kappa} s_{m,t} + q_t - D_t \right)^+ \right) \right. \\ & \quad \left. + \beta g^{(t+1)} \left( \left( \sum_{m=1}^{\kappa} s_{m,t} + q_t - D_t \right)^+ \right) \right\}. \quad (25) \end{aligned}$$

Next, we define the following new functions:

$$f_1^{(t)}(s_{\kappa,t}) \doteq -w \hat{\alpha}_{\kappa} s_{\kappa,t} - \pi E(D_t) \quad \text{and} \quad (26)$$

for each  $i = 2, \dots, \kappa$ ,

$$\begin{aligned} f_i^{(t)} \left( \sum_{m=\kappa-i+1}^{\kappa} s_{m,t} \right) \doteq & -w \hat{\alpha}_{\kappa-i+1} \left( \sum_{m=\kappa-i+1}^{\kappa} s_{m,t} \right) \\ & + \beta E \left\{ f_{i-1}^{(t+1)} \left[ \left( \sum_{m=\kappa-i+1}^{\kappa} s_{m,t} - D_t \right)^+ \right] \right\}. \quad (27) \end{aligned}$$

From parts (b) and (e) of Lemma 1, we can conclude that the functions  $\beta E \{ f_i^{(t+1)} (\sum_{m=\kappa-i}^{\kappa} s_{m,t} - D_t)^+ \} \in \mathcal{F}$ . Moreover, functions  $-w \hat{\alpha}_i (\sum_{m=i}^{\kappa} s_{m,t})$ , being decreasing linear functions of  $(\sum_{m=i}^{\kappa} s_{m,t})$ , also belong to  $\mathcal{F}$ . Finally, because the set  $\mathcal{F}$  is closed under summation, each  $f_i^{(t)} \in \mathcal{F}$ .

We now turn to the terms inside the maximization operator in Equation (25). In these arguments, we use  $s_t = (\sum_{m=1}^{\kappa} s_{m,t})$  to reduce notational burden. For reasons similar to those behind item (c) of Lemma 1, it can be proved that  $\hat{g}^{(t)}(s_t + q_t) \doteq E \{ (p + \pi) [(s_t + q_t) \wedge D_t] + \beta g^{(t+1)} [(s_t + q_t - D_t)^+] \}$  belongs to the set  $\mathcal{G}$ . It is also straightforward to see that

$E[-h(s_t + q_t - D_t)^+]$  is decreasing and has monotone decreasing differences in  $s_t + q_t$ . Therefore, this function and

$$\hat{f}_{\kappa}^{(t)}(s_t + q_t) = E \{ -h(s_t + q_t - D_t)^+ + \beta f_{\kappa}^{(t+1)} ((s_t + q_t - D_t)^+) \}$$

also belong to  $\mathcal{F}$ . The previous statement uses parts (b) and (e) of Lemma 1. We define a new function  $\bar{g}^{(t)}(s_t + q_t) \doteq \hat{g}^{(t)}(s_t + q_t) + \hat{f}_{\kappa}^{(t)}(s_t + q_t)$ . Then, from part (d) of Lemma 1, the function  $\bar{g}^{(t)}$  belongs to the set  $\mathcal{G}$ . Finally, let  $g^{(t)}(s_t) \doteq \max_{q_t \geq 0} \bar{g}^{(t)}(s_t + q_t)$ . Then, from part (a) of Lemma 1,  $g^{(t)}(s_t)$  belongs to the set  $\mathcal{G}$ . Now, from Equations (25)–(27) and the arguments above, we obtain Equation (13). Hence Proposition 1 is proved.  $\square$

PROOF OF THEOREM 1. Theorem 1 states that the retailer orders enough material in each period to bring its total inventory level to a critical level  $y_t^*$ . If  $s_t \geq y_t^*$ , then the order quantity is zero. Proof of Theorem 1 follows from observing that Equation (13) can be written alternatively as:

$$\begin{aligned} v_t(\tilde{s}_t) = & f_1^{(t)}(s_{\kappa,t}) + f_2^{(t)}(s_{\kappa,t} + s_{\kappa-1,t}) \\ & + \dots + f_{\kappa}^{(t)} \left( \sum_{m=1}^{\kappa} s_{m,t} \right) + \max_{a_t \geq s_t} \{ \bar{g}^{(t)}(a_t) \}, \quad (28) \end{aligned}$$

where  $\bar{g}^{(t)} \in \mathcal{G}$  and consequently has monotone decreasing differences. The base-stock level  $y_t^*$  is an unconstrained maximum of the function  $\bar{g}^{(t)}(\cdot)$ . (Note that when there are multiple optima, we choose the largest value.) Clearly, an optimal action is to order up to  $y_t^*$  if the current inventory level is below  $y_t^*$ . If not, ordering more lowers the value function and cannot be an optimal action.

It remains to show that  $y_t^* < \infty$ . This can be accomplished via an inductive argument. From the proof of Proposition 1,  $v_n(\tilde{s}_n)$  is decreasing in  $s_n$  (since  $g^{(n)} = 0$ ). Therefore,  $y_n^* = 0 < \infty$ . Let  $y_{t+1}^* < \infty$  be the inductive hypothesis. From (25),  $q_t^* = \arg \max_{q_t \geq 0} \bar{g}^{(t)}(s_t + q_t)$ , where

$$\begin{aligned} \bar{g}^{(t)}(s) = & E \{ (p + \pi) [s \wedge D_t] - h(s - D_t)^+ \\ & + \beta f_{\kappa}^{(t+1)} ((s - D_t)^+) + \beta g^{(t+1)} ((s - D_t)^+) \}. \quad (29) \end{aligned}$$

Next, the finiteness of  $E(D_t)$  ensures that  $P(D_t \leq s) \rightarrow 1$  and  $(s - D_t)^+ \rightarrow (s - D_t)$  as  $s \rightarrow \infty$ . Therefore, for a sufficiently large  $s$ :  $y_{t+1}^* \leq s < \infty$ , we have

$$\begin{aligned} \bar{g}^{(t)+}(s) = & (p + \pi) - (p + \pi + h) P(D_t \leq s) \\ & + \beta f_{\kappa}^{(t+1)+} ((s - D_t)^+) + \beta g^{(t+1)+} ((s - D_t)^+) \\ & < (p + \pi) - (p + \pi + h) P(D_t \leq s) < 0. \quad (30) \end{aligned}$$

The first inequality comes from the fact that  $f^{(t+1)+}(s)$  is nonpositive for any  $s$  and  $g^{(t+1)+}(s)$  is strictly negative for  $s \geq y_{t+1}^*$ . The final inequality is the consequence of finiteness of  $E(D_t)$  and the fact that  $h > 0$ . In summary, because the function  $\bar{g}^{(t)}(s)$  is decreasing when  $s$  is larger than a finite number,  $y_t^*$  is finite. Hence Theorem 1 is proved.  $\square$

PROOF OF PROPOSITION 3. Consider the arrival instance of a tagged replenishment. This moment is precisely  $\mu$  time



units after a demand arrival epoch that triggers the order. Because the Poisson process remains invariant under uniform translations, the replenishment arrival epochs also constitute a Poisson process (see Cox and Isham 1980 for details). Next, it follows from the Poisson-Arrivals-See-Time-Averages (PASTA) property that the shelf age distribution of the tagged replenishment is also the arbitrary time shelf age distribution of an item. Let  $IN^-(y)$  denote the stationary distribution of on-hand inventory at the moment of a replenishment arrival, but not counting the arriving item. Then

$$IN^-(y) = y - 1 - D(\mu), \quad (31)$$

and the arriving item observes the states  $\{\dots, -1, 0, 1, \dots, y - 1\}$  of  $IN^-(y)$ . Equation (31) is closely related to (15), the difference being that the arriving replenishment will observe no more than  $y - 1$  units in inventory.

If  $IN^-(y) \geq 0$ , then the arriving item must remain on the shelf for  $IN^-(y) + 1$  demand arrivals before it is consumed by a demand. However, if  $IN^-(y) < 0$ , then this replenishment is earmarked for a backordered demand and it is consumed immediately upon arrival. Let  $N_y$ , parameterized by the base-stock level  $y$ , be a random variable defined as follows:

$$N_y = \begin{cases} 0, & \text{if } IN^-(y) < 0, \\ IN^-(y) + 1, & \text{otherwise.} \end{cases} \quad (32)$$

With this definition in hand, it is clear that the random age of an arbitrary item is

$$A(y) = \sum_{i=1}^{N_y} X_i, \quad (33)$$

where  $X_i$  is the  $i$ th interarrival time and the empty sum is set to zero. From (31) and (32),  $P(N_y = x) = P(D(\mu) = y - x)$ , if  $x \in \{1, \dots, y\}$ , and  $P(N_y = 0) = P(D(\mu) > y - 1)$ , which leads to  $E[A(y)] = E[N_y]E[X_i] = E[(y - D(\mu))^+] / \lambda$ .

Given  $N_y = m$  and  $m > 0$ , the conditional distribution of  $A(y)$  is Erlang since it is a sum of  $m$  independent exponentially distributed interarrival times. Let  $G_m(t)$  denote the conditional CDF of  $A(y)$ . Then for  $m > 0$ ,  $G_m(t)$  can be written as follows:

$$G_m(t) = P(A(y) \leq t \mid N_y = m) = 1 - \sum_{i=0}^{m-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}. \quad (34)$$

Equation (19) is now obtained by unconditioning. That is,  $F_{A(y)}(t) = 1 \cdot P(N_y = 0) + \sum_{m=1}^y P(N_y = m)G_m(t)$ , which simplifies to (19) after substituting from above.  $\square$

**PROOF OF PROPOSITION 4.** From (33), we observe that  $A(y)$  is a random sum of i.i.d. random variables. In fact, since  $N_y$  and  $X_i$ s are mutually independent, it follows that  $E[a(A(y))] = E[E(a(A(y)) \mid N_y = m)]$ . Let  $\hat{a}(m) \doteq E[a(A(y)) \mid N_y = m]$ . We first claim that  $\hat{a}(m)$  is increasing and convex in  $m$ . This can be proved as follows.

Let  $X_i$  denote the  $i$ th demand interarrival time starting from the moment a tagged replenishment arrives that finds  $(m - 1)$  items in inventory on arrival, and let  $A_m$  denote this tagged item's shelf age. Then  $A_{m+1} = \sum_{i=1}^{m+1} X_i \geq_{st} A_m = \sum_{i=1}^m X_i$ . The notation " $\geq_{st}$ " stands for stochastically larger. If  $A$  and  $B$  are random variables and  $Ef(A) \geq Ef(B)$  for all nondecreasing functions  $f$  for which the expectations exist, then  $A$  is stochastically larger than  $B$  and is written  $A \geq_{st} B$ . An equivalent intuitive condition is that  $P(A > x) \geq P(B > x)$  for all  $x$  (see, for example, Müller and Stoyan 2002, for complete details). The claim  $A_{m+1} \geq_{st} A_m$  follows easily from the fact that  $X_i \geq 0$ . Also, since  $a(\cdot)$  is an increasing function, the above definition of stochastically larger implies that  $\hat{a}(m)$  is increasing in  $m$ .

Given  $A_{m-1}$ ,  $X_m$ , and  $X_{m+1}$ , only one of the following two possibilities exists: either  $A_{m-1} + X_m + X_{m+1} \geq A_{m-1} + X_m \geq A_{m-1} + X_{m+1} \geq A_{m-1}$  or  $A_{m-1} + X_m + X_{m+1} \geq A_{m-1} + X_{m+1} \geq A_{m-1} + X_m \geq A_{m-1}$ . Focusing on the first case, we can easily recognize that the following inequality must hold almost surely since  $a(\cdot)$  is convex in its argument (has increasing differences).

$$\begin{aligned} a(A_{m-1} + X_m + X_{m+1}) - a(A_{m-1} + X_m) \\ \geq a(A_{m-1} + X_{m+1}) - a(A_{m-1}) \end{aligned} \quad (35)$$

Similarly, if we assume that the inequality in the second case above holds, then it can be argued that

$$\begin{aligned} a(A_{m-1} + X_m + X_{m+1}) - a(A_{m-1} + X_{m+1}) \\ \geq a(A_{m-1} + X_m) - a(A_{m-1}) \end{aligned} \quad (36)$$

must hold almost surely. Taking expectations on both sides of the above inequalities and recognizing that  $E[a(A_{m-1} + X_m + X_{m+1})] = \hat{a}(m + 1)$ ,  $E[a(A_{m-1})] = \hat{a}(m - 1)$  and that  $E[a(A_{m-1} + X_{m+1})] = E[a(A_{m-1} + X_m)] = \hat{a}(m)$ , we obtain the following inequality from either of the above two cases:

$$\hat{a}(m + 1) + \hat{a}(m - 1) \geq 2\hat{a}(m). \quad (37)$$

This immediately completes the proof that  $\hat{a}(m)$  is convex in  $m$ . (The above inequality can be obtained by first rearranging the two earlier inequalities to yield a common inequality, which is  $a(A_{m-1} + X_m + X_{m+1}) + a_m(A_{m-1}) \geq a_m(A_{m-1} + X_m) + a_m(A_{m-1} + X_{m+1})$ , and then taking expectations on both sides.)

From the definition of  $\phi(y)$  in the statement of Proposition 4, we obtain

$$\begin{aligned} \phi(y) &= \sum_{m=0}^y \hat{a}(m)P(N_y = m) \\ &= \hat{a}(0)P(D(\mu) > y - 1) + \sum_{m=1}^y \hat{a}(m)P(D(\mu) = y - m). \end{aligned}$$

Using this definition, we now take differences and simplify as follows:

$$\begin{aligned}
 \phi(y+1) - \phi(y) &= \hat{a}(0)[P(D(\mu) > y) - P(D(\mu) > y-1)] \\
 &\quad + \sum_{j=1}^{y+1} \hat{a}(j)P(D(\mu) = y+1-j) - \sum_{j=1}^y \hat{a}(j)P(D(\mu) = y-j) \\
 &= \hat{a}(0)[P(D(\mu) = y)] + \sum_{i=0}^y \hat{a}(i+1)P(D(\mu) = y-i) \\
 &\quad - \sum_{j=1}^y \hat{a}(j)P(D(\mu) = y-j) \\
 &= \sum_{i=0}^y [\hat{a}(i+1) - \hat{a}(i)]P(D(\mu) = y-i). \tag{38}
 \end{aligned}$$

Clearly,  $\phi(y+1) - \phi(y) \geq 0$  since  $\hat{a}(\cdot)$  is increasing. A similar set of arguments leads to the following relationship:

$$\phi(y) - \phi(y-1) = \sum_{i=1}^y [\hat{a}(i) - \hat{a}(i-1)]P(D(\mu) = y-i) \tag{39}$$

On examining (38) and (39), it is clear that  $\phi(y+1) - \phi(y) \geq \phi(y) - \phi(y-1)$  for all  $y \geq 1$  because the function  $\hat{a}(m)$  is increasing convex in  $m$ .  $\square$

PROOF OF PROPOSITION 5. Define  $\phi_{\alpha}(y) \doteq E[a(A(y))]$  when credit terms are specified by  $\alpha$  and

$$\Delta_{\alpha}(y) \doteq \Pi_C(y | \alpha) - \Pi_C(y-1 | \alpha).$$

In (17), since  $E[I(y)]$  and  $E[B(y)]$  do not depend on  $\alpha$ , we have

$$\begin{aligned}
 \Delta_{\alpha_1}(y) - \Delta_{\alpha_2}(y) &= \lambda w \{ [\phi_{\alpha_1}(y) - \phi_{\alpha_1}(y-1)] - [\phi_{\alpha_2}(y) - \phi_{\alpha_2}(y-1)] \}.
 \end{aligned}$$

Recall from (39) that  $\hat{a}(i) = E[a(A(y)) | N_y = i]$  and therefore for a given sequence of interarrival times  $\{x_i\}$ ,  $[\hat{a}(i) - \hat{a}(i-1) | \{x_i\}] = \int_{\sum_{m=1}^{i-1} x_m}^{\sum_{m=1}^i x_m} \alpha(t) dt$ . Next,  $[\hat{a}(i) - \hat{a}(i-1) | \alpha = \alpha_1] \leq [\hat{a}(i) - \hat{a}(i-1) | \alpha = \alpha_2]$  since  $\alpha_1(t) \leq \alpha_2(t)$  for all  $t$ . This implies  $\phi_{\alpha_1}(y) - \phi_{\alpha_1}(y-1) \leq \phi_{\alpha_2}(y) - \phi_{\alpha_2}(y-1)$  and  $\Delta_{\alpha_1}(y) \leq \Delta_{\alpha_2}(y)$ . Therefore,  $\Pi_C(y | \alpha)$  is submodular in  $(y, -\alpha)$ . Recall that  $y_R(\alpha) = \arg \min_{y \geq 0} \{\Pi_C(y | \alpha)\}$ . Finally, using Theorem 8-4 in Heyman and Sobel (1984), it can be argued that  $y_R(\alpha)$  is increasing in  $-\alpha$ . Put differently,  $y_R(\alpha_1) \geq y_R(\alpha_2)$  when  $\alpha_1 \leq \alpha_2$ .  $\square$

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